PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Semiclassical Dispersion Theory of Lasers

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SEMICLASSICAL DISPERSION THEORY OF LASERS

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CONTENTS

This present paper is an attempt to describe the operation of a laser by a method patterned closely after the classical theory of dispersion. There, the electromagnetic field is treated as a classical system interacting with a collection of classical harmonic oscillators which are at rest. In the present case the radiation field is still treated classically, but, in accordance with the correspondence principle, the oscillators are now virtual oscillators associated with the up-and-down transitions in effective two-state atoms which are described quantum mechanically. The atoms carrying these virtual oscillators are not at rest but move with different velocities. We show how the Lorentz averaging can be performed for such systems, and derive a closed set of equations linking the average electric field **E**, with the average polarization density $\mathbf{P}_v$ and the average population inversion density m_v associated with atoms moving with the velocity **v**. Next, we investigate the conditions for the existence of a steady state if pumping is present, and if the electric field is represented by a standing wave, or a travelling wave of a single frequency (single-mode case). It turns out that, notwithstanding the nonlinear nature of the equations, the steady-state conditions give a simple complex dispersion relation; moreover, the real and imaginary part of this dispersion relation are equivalent to the usual heuristic expressions which specify the operating frequency in terms of the index of refraction and cavity length, and which balance the gain against the losses. The single-mode case is analysed in detail without any smallness assumptions for the resultant intensity. The small-intensity case gives the usual results exhibiting the tuning dip. For high intensities the relative depth of the dip tends to zero. At the end of the paper we discuss proposed extensions and additional applications.

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In principle the theory of a laser is a chapter in dispersion theory where both the radiation field and matter should be treated quantum mechanically. One believes that for intense radiation fields, it is sufficient to treat the field classically, and matter quantum mechanically.

We already know from the classical theory of dispersion (Born 1933) that considerable care must be exercised, since the final simplicity of the results arises only after one or more judicious averaging has been performed (different averages discussed by Hoek (1939)). Additional difficulties occur if the system dissipates, since then provision must be made in a microscopic theory to accommodate this fact. This gives rise to additional averaging and conceptual complications associated with the usual paraphernalia of statistical mechanics.

We shall discuss the usual Lorentz-averaging procedure in more detail than usual for the following reason. In any discussion of lasers one finally comes to grips with two sets of equations, though they may appear in different disguises. One is Maxwell's equations for the electric field driven by the atomic dipoles; the other is the time-dependent Schrödinger equation for the atomic dipoles in the presence of an electric field. The two electric fields appearing in the two equations are considered to be the same, and usually it is represented by a few Fourier modes in space and time. We are then faced with the following conceptual predicament. If this electric field is the actual, unaveraged, microscopic electric field, as required by Schrödinger's equation, the electric field will be a function undergoing wild variations in space and time in the neighbourhood of the dipoles, and the polarization densities appearing in Maxwell's equation will be discontinuous functions; thus neither of them can be represented by a few Fourier modes. If, on the other hand, we believe that the electric field is the Lorentz averaged field, why should it appear in Schrödinger's equation? Moreover, at the densities encountered in a laser, will a Lorentz averaging smooth out the polarization densities sufficiently to enable one to represent the polarization density as one or two simple Fourier modes? The simplest way to resolve these problems is to re-do the Lorentz averaging from the beginning; this is also essential, since in the present case the dipoles are not all at rest, nor are they all moving with the same uniform velocity, the two cases for which the averaging process is usually derived.

1. Derivation of the basic equations

(*a*) *Nature of the model*

Our system consists of a collection of atoms in a cavity, interacting with a radiation field, and, through the radiation field, with each other. There shall be no other direct interaction, such as collisions, between them. These atoms can move and an external mechanism maintains a fixed velocity distribution. There is no interaction between the translational and internal degrees of freedom; thus in a transition the atom does not suffer a recoil. There is another external mechanism, the pumping, which can transfer energy to the atoms raising their internal energy. This mechanism we shall only incompletely specify by telling how many atoms are produced in each laser level on the average per unit time.

By virtue of the fact that these atoms are in a cavity transitions between certain energy levels of these atoms can be of much greater importance than some others. In particular in a steady state those transitions will predominate for which there is an inversion and for

which the emitted or absorbed light corresponds to one of the low-loss resonance modes of the cavity. Hence, for the calculation of the emission and the absorption of radiation we think of the atom as a two-state system with a broadened upper state a and a broadened lower state b, with energies E_a and E_b such that $(E_a - E_b)/\hbar \approx \omega_0$, the frequency of the absorbed or emitted radiation. For the computation of the broadening and the population densities of the two levels, we take into account all transitions in and out of these states. The transitions out of the levels give rise to damping coefficients γ_a and γ_b which are a measure of the natural line width of the levels. One might refer to this description as an 'effective two-state atom' model.

In the present approximation the system is described by Maxwell's equations for the microscopic electromagnetic fields, driven by the microscopic polarization currents which arise from the transitions in the atoms. The transitions, in turn, are described by the time-dependent Schrödinger equation. These equations, however, are of tremendous complexity. In addition, they are reversible, and their solution depends on the initial data. There are several competing methods developed in the theory of irreversible processes to overcome this difficulty. Instead of using any of them, we follow the time-honoured pattern of dispersion theory.

Let us consider classically, for the moment, oscillating dipoles embedded in the moving atoms. The moving atoms are combined mentally into streams of a given velocity. Since the atoms do not change their velocities, the population of each stream is permanently fixed, and knowing the initial density distribution within each stream, one can find immediately its density distribution for any later time. The dipoles are coupled to the electromagnetic field according to the Newtonian equations of motion. Our first task is to develop the Lorentz averaging procedure for this collection of streaming dipoles. Once this is done each dipole stream will give rise to a polarization density stream; the latter generates a current, which appears as a source in Maxwell's equations. We shall now presumably have a closed set of equations for the Lorentz averaged fields and the polarization density of each stream, whose solutions depend on the initial values of the Lorentz averaged fields at each point, the initial value of the polarization density for each stream at each point, and the initial density distribution within each stream.

At this stage there is still no dissipation. (The appearance of radiation damping terms in the polarization density equation does not, of course, mean an actual damping; it is simply part of the coupling between the polarization density and the electromagnetic field, enabling an energy transfer from the polarization *to* the field.) Dissipation can now be provided by the introduction of an extraneous current which drains off energy according to Ohm's law. Additional dissipation and noise can be introduced similarly by a fluctuating current or electric field (Lorentz 1916; Landau & Lifshitz 1960). These currents may be present at the boundaries, within the system, or both. The introduction of this dissipation has as its consequence the obliteration of the initial data dependence of the solution for long times, except for its dependence on the initial density distribution in each stream. The latter must survive since the motion of the atoms is decoupled from anything in the system. If we assume the initial density distribution to be spatially homogeneous in each stream, and the number of atoms in each stream of a given velocity proportional to the Maxwell distribution of velocities, the translational degrees of freedom will start and stay in thermal equilibrium.

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If matter is to be described quantum mechanically, the scheme remains the same. Either we use the Schrödinger equation to describe matter, or to emphasize the similarity with the classical situation, we use the correspondence principle. In accordance with the latter, we associate virtual oscillators with the transitions occurring between atomic states. These oscillators obey similar equations of motion as the classical ones, however, now the density of the oscillators associated with the transitions can vary in space and time and an additional equation is needed to specify that. The latter equation can be interpreted either as a balance for the individual energy level population densities, or, alternatively, as an expression of energy balance for induced emissions.

(*b*) *Classical situation*

We shall first derive the basic equations for the classical case. Then using the correspondence principle we convert the set to cover the quantum mechanical description.

Consider first one dipole in motion. The electromagnetic field is described by the two fields **e** and **b** which has as its sources the microscopic polarization current and charge density due to the moving dipole. We represent the dipole as given in figure 1. $\mathbf{R}(t)$ is the

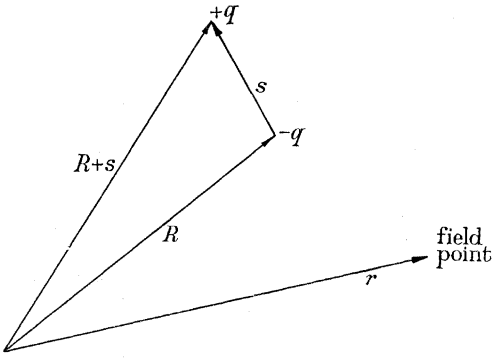

FIGURE 1. Elementary dipole with the coordinates of its constituents.

radius vector of the negative charge $-q$, $\mathbf{s}(t)$ is the radius vector of the positive charge relative to the negative one. Then the charge density at an arbitrary point $\mathbf{r}$ in space, outside the dipole is given by

$$\begin{aligned}\rho(\mathbf{r},t) &= q[\delta(\mathbf{R}+\mathbf{s}-\mathbf{r})-\delta(\mathbf{R}-\mathbf{r})] = -q(\mathbf{s}.\partial/\partial\mathbf{r})\,\delta(\mathbf{R}-\mathbf{r}) = -\operatorname{div} q\mathbf{s}\delta(\mathbf{R}-\mathbf{r})\\ &= -\operatorname{div}\mathbf{p}(\mathbf{r},t),\end{aligned} \tag{1}$$

where $q\mathbf{s}$ is the dipole moment, and $\mathbf{p}(\mathbf{r},t) = q\mathbf{s}\delta(\mathbf{R}-\mathbf{r})$, the microscopic polarization density due to the dipole. Similarly, the microscopic current density can be written as

$$\begin{aligned}\mathbf{j}(\mathbf{r},t) &= q\left[\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}+\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t}\right]\delta(\mathbf{R}+\mathbf{s}-\mathbf{r})-q\left(\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}\right)\delta(\mathbf{R}-\mathbf{r})\\ &= q\left(\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t}\right)\delta(\mathbf{R}-\mathbf{r})-q\left(\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}\right)(\mathbf{s}.\partial/\partial\mathbf{r})\delta(\mathbf{R}-\mathbf{r})\\ &= \frac{\partial\mathbf{p}(\mathbf{r},t)}{\partial t}+\operatorname{curl}(\mathbf{p}\times\mathbf{v});\end{aligned} \tag{2}$$

$\partial/\partial t$ means of course that $\mathbf{r}$ is being kept fixed in the differentiation; thus, as indicated, $\mathbf{R}(t)$ is attacked by this differentiation. The first term is well known; the second term, upon being averaged, brings in the so-called Röntgen current. Inserting these in Maxwell's equations we obtain

$$\left.\begin{aligned} \operatorname{curl}\mathbf{b} &= \frac{1}{c}\frac{\partial \mathbf{e}}{\partial t}+\frac{4\pi}{c}\frac{\partial \mathbf{p}}{\partial t}+\frac{4\pi}{c}\operatorname{curl}(\mathbf{p}\times\mathbf{v}),\\ \operatorname{curl}\mathbf{e} &= -\frac{1}{c}\frac{\partial \mathbf{b}}{\partial t},\\ \operatorname{div}\mathbf{e} &= -4\pi\operatorname{div}\mathbf{p},\\ \operatorname{div}\mathbf{b} &= 0; \end{aligned}\right\} \tag{3a}$$

or

$$\left.\begin{aligned} \operatorname{curl}\left[\mathbf{b}-\frac{4\pi}{c}(\mathbf{p}\times\mathbf{v})\right] &= \frac{1}{c}\frac{\partial(\mathbf{e}+4\pi\mathbf{p})}{\partial t},\\ \operatorname{curl}\mathbf{e} &= -\frac{1}{c}\frac{\partial \mathbf{b}}{\partial t},\\ \operatorname{div}(\mathbf{e}+4\pi\mathbf{p}) &= 0,\\ \operatorname{div}\mathbf{b} &= 0. \end{aligned}\right\} \tag{3b}$$

This set bears a strong formal resemblance to the macroscopic Maxwell's equations in the sense that $\mathbf{e}, \mathbf{e}+4\pi\mathbf{p}=\mathbf{d}; \mathbf{b}, \mathbf{b}-(4\pi/c)(\mathbf{p}\times\mathbf{v})=\mathbf{h}$ are in the same relation to each other as the macroscopic field vectors $\mathbf{E}, \mathbf{D}; \mathbf{B}, \mathbf{H}$. Of course in our case the quantities $\mathbf{d}$ and $\mathbf{h}$ are highly singular and only a space averaging makes them finite, though they remain discontinuous. At the moment they are a formal aid.

To close the equation we add explicitly the trajectory $\mathbf{R}(t)$ and the equations of motion for the internal motion of the dipole,

$$q\left[\frac{\mathrm{d}^2\mathbf{s}}{\mathrm{d}t^2}+(\omega_0^2+\gamma^2)\,\mathbf{s}+2\gamma\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t}\right]=\frac{q^2}{\mu}\left[\mathbf{e}'(\mathbf{R},t)+\frac{\mathbf{v}}{c}\times\mathbf{b}'(\mathbf{R},t)\right]. \tag{4}$$

ω_0 is the actual frequency in the presence of damping; thus it includes the frequency shift due to γ; μ is the mass of the dipole; $\mathbf{e}'[\mathbf{R}(t),t]$ and $\mathbf{b}'[\mathbf{R}(t),t]$ are the microscopic electric and magnetic fields at the point $\mathbf{r}=\mathbf{R}(t)$, excluding the field of the dipole itself, which is at the point $\mathbf{R}$. This contribution is already taken into account in the definition of μ and ω_0, and in the explicit introduction of the radiation damping force $-2\gamma\mu(\mathrm{d}\mathbf{s}/\mathrm{d}t)$ where γ represents the line width. This equation can be rewritten in terms of $\mathbf{p}(\mathbf{r},t)$ and $\mathbf{v}=\mathrm{d}\mathbf{R}/\mathrm{d}t$ as

$$\begin{aligned}\frac{\mathrm{D}_v^2\mathbf{p}}{\mathrm{D}t^2}+(\omega_0^2+\gamma^2)\,\mathbf{p}+2\gamma\frac{\mathrm{D}_v\mathbf{p}}{\mathrm{D}t} &= \frac{q^2}{\mu}\left[\mathbf{e}'(\mathbf{R},t)+\frac{\mathbf{v}}{c}\times\mathbf{b}'(\mathbf{R},t)\right]\delta(\mathbf{R}-\mathbf{r})\\ &= \frac{q^2}{\mu}\mathbf{f}(\mathbf{R},t)\,\delta(\mathbf{R}-\mathbf{r}),\end{aligned}$$

where $\mathrm{D}_v/\mathrm{D}t$ is the co-moving derivative $\partial/\partial t+\mathbf{v}.\partial/\partial\mathbf{r}$, and $\mathbf{f}$ is the abbreviation for the force appearing in the square bracket. (The equivalence immediately follows if we notice that $\mathrm{D}_v/\mathrm{D}t$ is the ordinary time derivative if it is applied to a function which is independent of $\mathbf{r}$, and gives zero if applied to functions, such as $\delta(\mathbf{R}-\mathbf{r})$, which combine $\mathbf{r}$ and t only in the combination $\mathbf{r}-\mathbf{R}(t)$.)

If we consider many dipoles, we label the polarization density due to each and sum over them. The summation we do in two steps. First we sum over all dipoles, which, at a given time, have the velocity $\mathbf{v}$, wherever they may be; this quantity we denote by $\mathbf{p}_v(\mathbf{r},t)$. Summing then over all velocities we obtain the total microscopic polarization

$$\sum_v \mathbf{p}_v(\mathbf{r},t) = \mathbf{p}(\mathbf{r},t).$$

This summation can immediately be performed on Maxwell's equations since v does not appear in the coefficients, only in the functions. Thus we can take over equations (3) as they stand, introducing the total microscopic electric and magnetic polarization densities,

$$\sum_v \mathbf{p}_v(\mathbf{r},t) \quad \text{and} \quad \sum_v \frac{\mathbf{p}_v \times \mathbf{v}}{c}.$$

A similar summation cannot be immediately performed on the differential equation for $\mathbf{p}_v$ to give an equation for $\mathbf{p}$ since the operator $\mathrm{D}_v/\mathrm{D}t$ is also affected by the summation. We can, however, perform the summation for all those dipoles which have a given velocity $\mathbf{v}$. This gives the following differential equation for $\mathbf{p}_v$:

$$\frac{\mathrm{D}_v^2 \mathbf{p}_v}{\mathrm{D}t^2} + (\omega_0^2+\gamma^2)\,\mathbf{p}_v + 2\gamma \frac{\mathrm{D}_v \mathbf{p}_v}{\mathrm{D}t} = \frac{q^2}{\mu} \sum_j \mathbf{f}_j(\mathbf{R}_j,t)\,\delta(\mathbf{R}_j-\mathbf{r}), \tag{5}$$

with $\dot{\mathbf{R}}_j = \mathbf{v}$. The summation on the right-hand side is over all those particles which at time t have the velocity $\mathbf{v}$. There will be one such equation for each value of $\mathbf{v}$. Maxwell's equations now describe the total field generated by all dipoles, while the polarization equations still contain reference to the individual electric fields, since the contribution coming from a given dipole has to be always omitted in the sum. The Lorentz averaging, however, enables us to close the set of equations. This average is defined as follows. Surround the point $\mathbf{r}$ with a sphere w which has a radius, say one tenth of a wavelength in size, and take within this volume the mean of the function to be averaged. The averaging can be approximately performed and we obtain the basic set of equations (see appendix A), Maxwell's equations, and

$$\frac{\mathrm{D}_v^2 \mathbf{P}_v}{\mathrm{D}t^2} + (\omega_0^2+\gamma^2)\,\mathbf{P}_v + 2\gamma \frac{\mathrm{D}_v \mathbf{P}_v}{\mathrm{D}t} = \frac{q^2}{\mu} n_v(\mathbf{r},t)\left[\mathbf{E}(\mathbf{r},t) + (\tfrac{4}{3}\pi)\,\mathbf{P} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right], \tag{6}$$

where $$\mathbf{P} = \sum_v \mathbf{P}_v(\mathbf{r},t) \quad \text{and} \quad n_v(\mathbf{r},t) = \frac{1}{w}\int_w \mathrm{d}^3\mathbf{r} \sum_{j(\dot{\mathbf{R}}_j=\mathbf{v})} \delta(\mathbf{R}_j-\mathbf{r}).$$

Capital letters refer, as usual, to Lorentz-averaged quantities. If the system is dense enough so that many dipoles are in the Lorentz sphere, n_v can be approximated by a continuous function, and then $\mathbf{P}_v$ and $\mathbf{P}$ will also be continuous. However, if the system is very dilute, n_v will be discontinuous, and so will $\mathbf{P}_v$ and $\mathbf{P}$, and only an additional averaging can provide continuity.

(c) *Quantum situation*

Assume now that the dipoles are quantum mechanical objects, and consider electric dipole transitions. In this case the Correspondence Principle enables us to reach the final equations with ease (Rubinowitz 1933; Wentzel 1933). Consider two atomic states a and

b, a being the one with the larger energy. In a down-transition the energy $\hbar\omega_0 = E_a - E_b$ is emitted; in an up-transition the same amount of energy is absorbed. Now we cannot make statements about the individual acts of absorption and emission, but we can only make statistical predictions. They will be as follows. We consider a given external electric field and an atom so massive that its transitional motion can be treated classically. This atom is immersed many times into the external electric field $\mathbf{e}'(\mathbf{r}, t)$ at a given point $\mathbf{r} = \mathbf{R}$. What will be the behaviour of the expectation value of the dipole moment $\mathfrak{D}$? One finds that this expectation value can be conceived as the sum of $\mathfrak{D}_{\text{up}}$ and $\mathfrak{D}_{\text{down}}$, each obeying the following differential equations

$$\left.\begin{aligned} \frac{\mathrm{d}^2\mathfrak{D}_{\text{up}}}{\mathrm{d}t^2} + (\omega_0^2+\gamma^2)\,\mathfrak{D}_{\text{up}} + 2\gamma\frac{\mathrm{d}\mathfrak{D}_{\text{up}}}{\mathrm{d}t} &= f_{\text{up}}\frac{q^2}{\mu}\,\mathbf{e}'(\mathbf{R}, t), \\ \frac{\mathrm{d}^2\mathfrak{D}_{\text{down}}}{\mathrm{d}t^2} + (\omega_0^2+\gamma^2)\,\mathfrak{D}_{\text{down}} + 2\gamma\frac{\mathrm{d}\mathfrak{D}_{\text{down}}}{\mathrm{d}t} &= f_{\text{down}}\frac{q^2}{\mu}\,\mathbf{e}'(\mathbf{R}, t), \end{aligned}\right\} \tag{7}$$

where q^2/μ is given by $2\wp^2\omega_0/\hbar$, $\wp$ being the matrix element of the dipole transition; f_{up} and $-f_{\text{down}}$ are the probabilities that the atom is in state b and a respectively. The γ term corresponds to the damping, and γ itself is the mean of the two γ's associated with the two levels.

Thus we may say that we associate virtual dipole oscillators with each transition, an up-oscillator, and a down-oscillator, which have the same properties as the classical dipole oscillators, except that the sign of the coupling is reversed for the down-oscillators. Let us imagine now a large number of these virtual oscillators present at points $\mathbf{R}_j$ as before. The only difference is that now the label j should also tell whether the oscillator is a down or an up oscillator. Appending this label j to f and $\mathfrak{D}$ in (7), the situation is now the same as in the classical case, and we may proceed precisely as before. We immediately reach the differential equations for $\mathbf{P}_v(\mathbf{r}, t)$.

$$\frac{\mathrm{D}_v^2\mathbf{P}_v}{\mathrm{D}t^2} + (\omega_0^2+\gamma^2)\,\mathbf{P}_v + 2\gamma\frac{\mathrm{D}_v\mathbf{P}_v}{\mathrm{D}t} = \frac{2\wp^2\omega_0}{\hbar}\,m_v\left[\mathbf{E}(\mathbf{r}, t) + \tfrac{4}{3}\pi\,\mathbf{P} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right], \tag{8}$$

with $$m_v(\mathbf{r}, t) = \sum_{j(\dot{\mathbf{R}}_j=\mathbf{v})}\frac{1}{w}\int f_j\,\delta(\mathbf{R}_j(t)-\mathbf{r})\,\mathrm{d}^3\mathbf{r} = n_v(\mathbf{r}, t; b) - n_v(\mathbf{r}, t; a).$$

This is the same as the classical result, except for the appearance of

$$m_v(\mathbf{r}, t) = n_v(\mathbf{r}, t;\ b) - n_v(\mathbf{r}, t;\ a)$$

in place of n_v. $n_v(\mathbf{r}, t;\ a)$ is the average density of atoms which are in state a at time t with the velocity $\mathbf{v}$, and are in the Lorentz sphere surrounding $\mathbf{r}$; m_v thus represents a population difference. If it is negative there is a population inversion at $\mathbf{r}$ for atoms moving with the velocity v. The sign has been chosen so that m positive corresponds to a preponderence of up-transitions, so that for lasing, m shall, as we surmise, turn out to be negative. In a dilute system m_v is small in absolute value; in fact, if we measure the volume in units of Lorentz spheres, we shall find m hardly ever more than zero, or plus, or minus one; for denser systems this will improve. The correction terms $\frac{4}{3}\pi\mathbf{P} + (\mathbf{v}/c)\times\mathbf{B}$ will often be small.

If m_v were held fixed by some means, we would have the same equation system as in the classical case. However m_v can change, and we must find out how. This is determined by the balance equation for the individual energy level population densities. For the moment, let

us consider the atoms at rest. Each population density will change for three reasons; spontaneous transitions out of the state, induced transitions, and pumping. Let us consider the upper state a: in time $\mathrm{d}t$ the change $\mathrm{d}n_a$ in n_a is given by

$$\mathrm{d}n_a = -\gamma_a n_a \mathrm{d}t + \mathbf{E}_{ac} \cdot \frac{\mathrm{d}\mathbf{P} + \gamma_a \mathbf{P}\,\mathrm{d}t}{\hbar\omega_0} + \Lambda_a \mathrm{d}t.$$

The first term represents the depletion due to spontaneous transitions into all other states; the second term is obtained if we notice that $\hbar\omega_0 \delta n_a$ is the energy density absorbed by the atoms (if δn_a, the change in n_a due to induced transitions, is positive) and $\mathbf{E}_{ac} \cdot [\mathrm{d}\mathbf{P} + \gamma \mathbf{P}\,\mathrm{d}t]$ is the work done per unit volume during $\mathrm{d}t$ by the actual electric field $\mathbf{E}_{ac}$ on the polarization $\mathbf{P}$ in induced changes, $\mathrm{d}\mathbf{P} - (-\gamma \mathbf{P}\,\mathrm{d}t)$ being the change in the polarization density due to induced emissions; Λ_a is the change in n_a per unit time due to pumping. For the lower level b we get a similar equation with γ_b and Λ_b appearing, except for two changes. The second term acquires the opposite sign. It now represents a negative work done, since δn_b positive implies an energy loss for the atom. Also, there is an additional term proportional to $n_a \mathrm{d}t$ which represents the population increase due to spontaneous decay out of level a. This latter term we shall neglect, since in a steady state it only alters (in general only slightly) the effective population inversion density. If γ_a is equal to γ_b is equal to γ, we get the following equation for $n_b - n_a = m$, the population density difference,

$$\frac{\mathrm{d}m}{\mathrm{d}t} + \gamma m = -\frac{2\mathbf{E}_{ac}}{\hbar\omega_0} \cdot \left(\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} + \gamma \mathbf{P} \right) + (\Lambda_b - \Lambda_a); \tag{9}$$

[(8) and (9) were used for atoms at rest by Grasiuk & Oraevskij (1964)]. Multiplying (9) by $\frac{1}{2}\hbar\omega_0$ we can also interpret this result as an energy balance in induced emissions. For, if an up-transition occurs, n_a changes by $+1$, and n_b changes by -1, hence m changes by -2, and the energy absorbed is $\hbar\omega_0 = -(\frac{1}{2}\hbar\omega_0)\,(-2)$. Thus if m changes by δm, $-(\frac{1}{2}\hbar\omega_0)\,\delta m$ is the quantity of energy absorbed. During a time $\mathrm{d}t$, $\mathrm{d}m - (-\gamma m\,\mathrm{d}t)$ is the change in m due to induced transitions; the last term being subtracted represents the spontaneous part. A quantity of energy $-(\frac{1}{2}\hbar\omega_0)\,(\mathrm{d}m + \gamma m\,\mathrm{d}t)$ will be absorbed per unit volume, which is provided by the work done by the actual electric field in induced transitions, $\mathbf{E}_{ac}$, and by the external pumping. The former is $\mathbf{E}_{ac} \cdot (\mathrm{d}\mathbf{P} + \gamma \mathbf{P}\,\mathrm{d}t)$; the pumping should contribute the amount $(\frac{1}{2}\hbar\omega_0)\,(\Lambda_b - \Lambda_a)\,\mathrm{d}t$.

If the atoms move, we immediately get in the laboratory frame a set of equations, one for each $\mathbf{v}$,

$$\frac{\mathrm{D}_v m_v}{\mathrm{D}t} + \gamma (m_v - m_v^{(0)}) = -\frac{2}{\hbar\omega_0} \left[\mathbf{E} + \tfrac{4}{3}\pi \mathbf{P} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right] \cdot \left(\frac{\mathrm{D}_v \mathbf{P}_v}{\mathrm{D}t} + \gamma \mathbf{P} \right), \tag{10}$$

where the actual electric field in the rest frame $\mathbf{E}_{ac} = \mathbf{E} + \frac{4}{3}\pi \mathbf{P}$ acquires the additional term $(\mathbf{v}/c) \times \mathbf{B}$. Here we replaced $\Lambda_b - \Lambda_a$ by an externally produced population density supply per unit time, denoted as $\gamma m_v^{(0)}$, for convenience.

If we consider the widths of the two levels different, we must use the n_a and n_b equations separately instead of their difference. One can also work out everything for this case following stepwise the simpler procedure. In the final results we indicate the trivial differences.

An alternative derivation of (7) and (9) can be provided starting from quantum theory. This is indicated in appendix B.

2. The basic equations

Maxwell's equation are given as

$$\left.\begin{aligned}
\operatorname{curl}(\mathbf{B}-4\pi\mathbf{Q}) &= \frac{1}{c}\frac{\partial(\mathbf{E}+4\pi\mathbf{P})}{\partial t}+\frac{4\pi}{c}\mathbf{j}_{\text{ex.}},\\
\operatorname{curl}\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},\\
\operatorname{div}\mathbf{B} &= 0,\\
\operatorname{div}(\mathbf{E}+4\pi\mathbf{P}) &= 4\pi\rho_{\text{ex.}},\\
\mathbf{Q} &= \sum_v \mathbf{P}_v\times\frac{\mathbf{v}}{c},\\
\mathbf{P} &= \sum_v \mathbf{P}_v,\\
\frac{\partial\rho_{\text{ex.}}}{\partial t}+\operatorname{div}\mathbf{j}_{\text{ex.}} &= 0,\\
\mathbf{j}_{\text{ex.}} &= \left(\frac{1}{4\pi}\right)2\sigma\mathbf{E}.
\end{aligned}\right\} \tag{11}$$

The external charge $\rho_{\text{ex.}}$, and the external current $\mathbf{j}_{\text{ex.}}$, are responsible for the ohmic losses inside the cavity. For future convenience we take $2\sigma/4\pi$ to denote the standard conductivity.

If we assume $\rho_{\text{ex.}}$ to be zero, then, from the continuity equation, Ohm's law, and the divergence equation on $\mathbf{E}+4\pi\mathbf{P}$, it immediately follows that the divergences of $\mathbf{P}$ and $\mathbf{E}$ are both zero.

If we separate the fields in Maxwell's equations we obtain

$$\begin{aligned}
\frac{\partial^2\mathbf{E}}{\partial t^2}+2\sigma\frac{\partial\mathbf{E}}{\partial t}-c^2\Delta\mathbf{E} &= -4\pi\frac{\partial}{\partial t}\left(\sum_v\frac{\partial\mathbf{P}_v}{\partial t}+\operatorname{curl}\mathbf{P}_v\times\mathbf{v}\right)\\
&= -4\pi\frac{\partial}{\partial t}\sum_v\frac{\mathrm{D}_v\mathbf{P}_v}{\mathrm{D}t},
\end{aligned} \tag{12}$$

$$\left.\begin{aligned}
\operatorname{div}\mathbf{E} &= 0;\\
\frac{\partial^2\mathbf{B}}{\partial t^2}-c^2\Delta\mathbf{B} &= 4\pi c\operatorname{curl}\sum_v\frac{\mathrm{D}_v\mathbf{P}_v}{\mathrm{D}t},\\
\operatorname{div}\mathbf{B} &= 0.
\end{aligned}\right\} \tag{13}$$

The last step in (12) follows from the well-known identity for any vector field $\mathbf{F}$, and velocity field $\mathbf{v}$ independent of $\mathbf{r}$,

$$\frac{\mathrm{D}_v\mathbf{F}}{\mathrm{D}t}=\frac{\partial\mathbf{F}}{\partial t}+\operatorname{curl}(\mathbf{F}\times\mathbf{v})+\mathbf{v}\operatorname{div}\mathbf{F},$$

with $\operatorname{div}\mathbf{F}=0$.

From (12) and (13) we see that the source of the fields is the current which is generated by stimulated and spontaneous processes. We shall restrict our interest to that part of the electromagnetic field which arises through induced processes. We then replace the total polarization current

$$\sum_v\frac{\mathrm{D}_v\mathbf{P}_v}{\mathrm{D}t}\quad\text{by}\quad\sum_v\frac{\mathrm{D}_v\mathbf{P}_v}{\mathrm{D}t}-(-\gamma\mathbf{P}_v),$$

the last term being the current generated by spontaneous emissions. In the future we shall denote only this portion of the electric magnetic field by $\mathbf{E}$ and $\mathbf{B}$. The introduction of

$\mathrm{D}_v\mathbf{P}/\mathrm{D}t+\gamma\mathbf{P}_v$ into the $\mathbf{E}$ equation is motivated by two reasons: first, it leads to a mathematical simplification, since then, as we shall see, only this combination of $\mathbf{P}_v$ and its derivative plays a role in our equations; secondly, this step is numerically permitted, since we know that in the steady-state operation of a laser, the contribution of spontaneous emission to the $\mathbf{E}$ field in a given mode is negligible, except very close to threshold.

Our basic differential equation system is then given by

$$\frac{\partial^2\mathbf{E}}{\partial t^2}+2\sigma\frac{\partial\mathbf{E}}{\partial t}-c^2\Delta\mathbf{E}=-4\pi\frac{\partial}{\partial t}\left(\sum_v\frac{\mathrm{D}_v\mathbf{P}_v}{\mathrm{D}t}+\gamma\mathbf{P}_v\right),$$
$$\operatorname{div}\mathbf{E}=0; \tag{14}$$

$$\left.\begin{aligned}\frac{\partial^2\mathbf{B}}{\partial t^2}-c^2\Delta\mathbf{B}&=4\pi c\operatorname{curl}\left(\sum_v\frac{\mathrm{D}_v\mathbf{P}_v}{\mathrm{D}t}+\gamma\mathbf{P}_v\right),\\ \operatorname{div}\mathbf{B}&=0;\end{aligned}\right\} \tag{15}$$

$$\frac{\mathrm{D}_v^2\mathbf{P}_v}{\mathrm{D}t^2}+2\gamma\frac{\mathrm{D}_v\mathbf{P}_v}{\mathrm{D}t}+(\omega_0^2+\gamma^2)\,\mathbf{P}_v=\left(2\wp^2\frac{\omega_0}{\hbar}\right)m_v\mathbf{E}, \tag{16}$$

$$\frac{\mathrm{D}_v m_v}{\mathrm{D}t}+\gamma(m_v-m_v^{(0)})=-\frac{2}{\hbar\omega_0}\mathbf{E}\cdot\left(\frac{\mathrm{D}_v\mathbf{P}_v}{\mathrm{D}t}+\gamma\mathbf{P}_v\right). \tag{17}$$

(In (16) and (17) we have omitted in the force term both the polarization correction and the Lorentz force. The former is justifiable in moderately dilute gases. One would be tempted to neglect the Lorentz force arguing that it contains the small factor v/c. However, the Doppler shift, which we shall continue to include, contains the same small parameter. Notwithstanding, the omission of the Lorentz force is reasonable. In the results one can find that the two effects enter in a different fashion. The Doppler shift has a major effect on the resonances, so it has to be retained. On the other hand, the omission of the Lorentz force makes one miscompute the energy density of the electric field responsible for induced emissions, occurring in a moving atom, by replacing the energy density as seen by the moving atom with that seen by the same atom at the same place, but at rest.)

Given $m_v^{(0)}(\mathbf{r},t)$, boundary conditions, and initial data we have a well-defined initial value problem. If we restrict our interest to stationary solutions, we can dispense with the initial data, since the assumed losses will obliterate them anyway. $m_v^{(0)}$ is a stochastic function of $\mathbf{r}$ and t; we shall assume its mean to be independent of these variables and Maxwellian in $\mathbf{v}$. If we disregard any correlation between m_v and the other quantities we can replace m_v by its stochastic mean in the differential equations. We shall do this, and use the same symbol, m_v, for this mean.

For the rest of the paper we shall consider a one-dimensional problem and envisage a long closed cavity of length L. To insure that the problem is properly one-dimensional we shall average all functions which depend on the position over planes perpendicular to the axis of the cavity. This averaging, coupled with the already performed Lorentz averaging over small spheres, is the same as a single Lorentz averaging over thin sections, perpendicular to the axis of the cavity, and of the width of a Lorentz sphere. This is not only useful, but also necessary. If the density of the active atoms is very dilute, as it may be in gaseous lasers, the averaging over a Lorentz sphere will not insure the continuity of the $\mathbf{P}$ and m fields. One is then prohibited from representing these functions, and the $\mathbf{E}$ field, as a combination of

a few spatial Fourier modes. Let us take m to be of the order of magnitude of the density, about $10^8/\mathrm{cm}^3$. For red light the volume of a Lorentz sphere is of the order of $10^{-16}/\mathrm{cm}^3$. Thus one must sample 10^8 Lorentz spheres to find one of them occupied. In fact, to get on the average, the meagre occupancy of a single atom per Lorentz sphere (which is far from being sufficient for continuity), one would need at least a density of $10^{16}/\mathrm{cm}^3$. This is above the usual densities encountered in gaseous lasers. On the other hand, for a cavity a few millimetres in diameter we have about 10^9 Lorentz spheres per cross-section, which gives $10^7/\mathrm{cm}^3$ for the density at which one finds one atom per averaging volume.

The boundary conditions are as follows. We simulate the boundary losses by the ohmic dissipation inside the cavity, and hence assume $\mathbf{E}$ to vanish at the boundaries and outside. (This will certainly give an improper amplitude dependence of the $\mathbf{E}$ field inside the cavity, but simplifies the problem. At some increase of complication one is able to discuss the case of no bulk loss but surface losses. The solution then exhibits the expected amplitude variation of the field.) The quantities m_v and $\mathbf{P}_v$ should vanish on and outside the boundaries.

3. The solution for single-mode operation

(a) *General properties of the solution*

We do not intend to solve the complete initial value problem. Rather, we seek the conditions which have to be satisfied for a stationary single-mode operation. By single mode we mean that the electric field can be described by one standing wave with a wave vector of magnitude $k = n\pi/L$, n being a large positive integer (k thus is *always* positive), and with a frequency $\omega(k)$ which is to be determined. Each $\mathbf{P}_v$ and $\mathbf{E}$ are parallel to each other, and perpendicular to the axis of the cavity, and will be functions of x and t, the x axis being along the cavity. The vector $\mathbf{v}$ has only one non-vanishing component, v_x, which we shall denote by v. Thus v carries a sign. If positive, the motion is along the positive x direction. Henceforth we omit all vector signs from the equations.

We assume E to be of the form $\mathscr{E}\sin kx\cos\omega t$, with ω and $\mathscr{E}$ unknown. Our problem is the following. Given k and $m^{(0)} = \sum_v m_v^{(0)}$, find those values of $\mathscr{E}$, (or $\mathscr{E}^2$ since the sign of $\mathscr{E}$ does not matter), and ω for which such a solution is possible. Thus we seek two functions $\mathscr{E}^2(m^{(0)}, k)$ and $\omega(m^{(0)}, k)$ or alternatively two other functions $\omega(\mathscr{E}^2, k)$ and $\mathscr{E}^2(m^{(0)}, \omega)$. The first function corresponds to a dispersion relation which is intensity dependent, the second gives the intensity as a function of the frequency and the pumping. The usual dispersion theory arises as a limit for $\mathscr{E}^2$ tending to zero, in which case the second equation becomes inoperative, and the first gives the usual intensity–independent dispersion relation.

The *modus operandi* is as follows. We insert the assumed form of E into the P_v and m_v equations and disregard temporarily the E equation. There are as many pairs of such equations as permitted v values, a pair for a given v value decoupled from all other pairs. Let us solve each pair, and find P_v, hence its sum, hence the source for E. Returning to the E equation we determine the conditions that the obtained source should generate an E of the assumed form. These conditions will be our goal. In practice some of these steps may be obviated, depending on the particular nature of the functions involved.

To proceed rapidly we do now two things. First, we transform the P_v equations into complex first-order differential equations; then we list a number of useful properties of the

solutions of these equations. Introduce the function $J_v = (\mathrm{D}_v/\mathrm{D}t+\gamma)\,P_v+\mathrm{i}\omega_0 P_v$, which, as it may be readily verified from (16), satisfies the differential equation

$$\frac{\mathrm{D}_v J_v}{\mathrm{D}t}-\mathrm{i}\omega_0 J_v+\gamma J_v = \frac{2\wp^2\omega_0}{\hbar} m_v E. \tag{18}$$

The real part of J_v, J_{Rv}, serves as the forcing function for both the m_v equation and the E equation. Thus, instead of P_v, it is sufficient to determine J_v. For each v we get a pair of first-order differential equations, (18) and

$$\frac{\mathrm{D}_v m_v}{\mathrm{D}t}+\gamma(m_v-m_v^{(0)}) = -\frac{2}{\hbar\omega_0} E J_{Rv}. \tag{19}$$

What are the general properties of the stationary solutions defined by these equations for the above E? We show in appendix C that: (*a*) the solutions are unique if they exist at all; (*b*) $\overline{m}_v^2 < \overline{m}_v m_v^{(0)}$, where the bar denotes a space average and time average; (*c*) $\overline{m}_v m_v^{(0)} > 0$; (*d*) $m_v(x,t)$ has a term independent of t, and negligibly small terms oscillating with frequences 2ω, 4ω, 6ω, etc. It thus may be taken to be a constant in time. (*e*) $m_v(x) = m_{-v}(-x)$; (*f*) $m_v(x)$ is periodic with the period of $E^2, \pi/k$. Thus $m_v(x)$ may be represented by a Fourier series

$$m_v(x) = \overline{m}_v + \sum_{n=1}^{\infty} a_n(v)\sin 2nkx + b_n(v)\cos 2nkx.$$

(*g*) J has the same x and t dependence as mE, the source term in the J equation. Thus, it oscillates primarily with the frequency ω and, at a given time, it may be represented by the series

$$J_v = \sum_{n=0}^{\infty} a_n'(v)\sin(2n+1)\,kx + b_n'(v)\cos(2n+1)\,kx.$$

Both the time and space averages of J_v are zero.

(*b*) *Dispersion relations*

The formal steady-state solutions of the J and m equations are

$$\left.\begin{aligned} J_{Rv}(x,t) &= \frac{2\wp^2\omega_0}{\hbar}\int_0^{\infty} \mathrm{d}\tau\, \mathrm{e}^{-\gamma\tau}\cos\omega_0\tau\, E(x-v\tau, t-\tau)\, m_v(x-v\tau),\\ m_v(x) &= m_v^{(0)} - \frac{2}{\hbar\omega_0}\int_0^{\infty}\mathrm{d}\tau\,\mathrm{e}^{-\gamma\tau} E(x-v\tau, t-\tau)\, J_{Rv}(x-v\tau, t-\tau), \end{aligned}\right\} \tag{20}$$

in which the independence of m_v on the time, as discussed above, has been noted. We now explicitly make use of the periodic nature of $m_v(x)$, property (*f*), by expressing it as a Fourier series

$$m_v(x) = \overline{m}_v + \sum_{n=1}^{\infty} a_n(v)\sin 2nkx + b_n(v)\cos 2nkx, \tag{21}$$

and go about computing the source in the E equation. It follows from the symmetry property of m_v, property (*e*) above, that $\overline{m} = \overline{m}_{-v}$, $a_n(v) = -a_n(-v)$, and $b_n(v) = b_n(-v)$. Thus each term in J_{Rv} can be identified at once as being either symmetric, or antisymmetric in v. The antisymmetric terms make no contribution to $\sum_v J_{Rv}$, and can be neglected at this point.

Similarly, in the τ integration, terms arise which are smaller than the others by a factor γ/ω. These too may be profitably neglected. Then, $\sum_v J_{Rv}$ is of the form

$$\left.\begin{aligned}\sum_v J_{Rv}(x,t) = \frac{\wp^2\omega_0}{\hbar}\mathscr{E}\sin kx \times \sum_v (\overline{m}_v - \tfrac{1}{2}b_1(v))\, L(\omega-kv)\,[\gamma\cos\omega t + (\omega-\omega_0-kv)\sin\omega t] \\ +\tfrac{1}{4}a_1(v)\,[L(\omega+kv)\,(\omega-\omega_0+kv) - L(\omega-kv)\,(\omega-\omega_0-kv)]\cos\omega t \\ -\tfrac{1}{4}a_1(v)\,[L(\omega+kv)-L(\omega-kv)]\,\gamma\sin\omega t + \text{terms with } x \text{ dependence} \\ \text{of the form } \exp(ink x),\ n \text{ being any odd integer except } \pm 1 \\ \text{(Property } (g) \text{ above)}\end{aligned}\right\} \quad (22)$$

where $$L(\omega) = [(\omega-\omega_0)^2+\gamma^2]^{-1}.$$

Because $\sum_v J_{Rv}$ has only an $\exp\{\pm i\omega t\}$ time dependence, these latter terms as sources in the wave equation make a negligibly small change in the assumed form for $E(x,t)$. We therefore do not retain them.

It is clear from (22) that a detailed knowledge of $m_v(x)$ is not needed to derive the dispersion relations, just a knowledge of $\overline{m}_v$, $a_1(v)$, and $b_1(v)$. But equations (20) do not permit a determination of all of the Fourier coefficients. Further simplifications are still possible, however.

Rewrite (20) as

$$m_v = m_v^{(0)} - \frac{4\wp^2}{\hbar^2}\int_0^\infty d\tau' \int_0^\infty d\tau \exp\{-\gamma(\tau+\tau')\}\cos\omega_0\tau \times E(x-v\tau', t-\tau')\, E[x-v(\tau+\tau'), t-(\tau+\tau')]\, m_v[x-v(\tau+\tau')]. \quad (23)$$

Using (21) for m_v, we carry out the indicated integration over τ and τ', and equate the x-independent terms. One obtains

$$\overline{m}_v = m_v^{(0)} - \frac{\wp^2\mathscr{E}^2}{4\hbar^2\gamma^2}\{(\overline{m}_v - \tfrac{1}{2}b_1(v))\,\gamma^2[L(\omega+kv)+L(\omega-kv)] + \tfrac{1}{2}a_1(v)\,\gamma[(\omega-\omega_0+kv)\,L(\omega+kv) + (\omega-\omega_0-kv)\,L(\omega-kv)]\}, \quad (24)$$

from which it follows that the quantity $\overline{m}_v - \frac{1}{2}b_1(v)$, appearing in (22), is

$$\overline{m}_v - \tfrac{1}{2}b_1(v) = \frac{m_v^{(0)} - \frac{1}{2}b_1(v) - \frac{1}{2}a_1(v)\,B\gamma\,[(\omega-\omega_0+kv)\,L(\omega+kv) - (\omega-\omega_0-kv)\,L(\omega-kv)]}{1+B\gamma^2\,[L(\omega+kv)+L(\omega-kv)]}, \quad (25)$$

where $B = \wp^2\mathscr{E}^2/4\hbar^2\gamma^2$ is a normalized intensity parameter.

We now show that the b_1 and a_1 terms in the numerator of (25) can be neglected compared to $m_v^{(0)}$. First we know that

$$\frac{1}{2}\sum_{n=1}^{\infty} a_n^2(v) + b_n^2(v) = \overline{m_v^2} - (\overline{m}_v)^2 \geqslant 0. \quad (26)$$

It follows from property (b) that

$$\frac{1}{2}\left[\sum_{n=1}^{\infty} a_n^2(v) + b_n^2(v)\right]^{\frac{1}{2}} < \frac{m_v^{(0)}}{\sqrt{2}}\left[\frac{\overline{m}_v}{m_v^{(0)}} - \left(\frac{\overline{m}_v}{m_v^{(0)}}\right)^2\right]^{\frac{1}{2}}. \quad (27)$$

Property (*c*) tells us that $\overline{m}_v/m_v^{(0)} > 0$. The maximum value of the r.h.s. of (27) is thus $0{\cdot}35\, m_v^{(0)}$, occurring when $\overline{m}_v/m_v^{(0)} = \frac{1}{2}$. We are able to conclude from this that $\frac{1}{2}a_1(v)$ and $\frac{1}{2}b_1(v)$ are each small compared to $m_v^{(0)}$.

To go further, we examine the exact solution of (23) for the two cases where an analytical solution is possible, $kv \gg \gamma$, and $kv = 0$. In the first case, $kv \gg \gamma$, all of the x-dependent terms on the r.h.s. of (23) are zero. We obtain as the solution, therefore, from (24) and (25)

$$m_v = \overline{m}_v = \frac{m_v^{(0)}}{1+B\gamma^2[L(\omega+kv)+L(\omega-kv)]}. \tag{28}$$

Here $a_1(v) = 0$ and $b_1(v) = 0$ so that neglecting the second and third terms in the numerator of (25) is certainly justified. When $kv = 0$, (23) yields

$$m_0(x) = \frac{m_v^{(0)}}{1+4B\gamma^2 L(\omega)\sin^2 kx}. \tag{29}$$

Now, $a_1(0) = 0$ and $\frac{1}{2}b_1(0) = K[1+2K+(K+1)\,(1+2K)^{\frac{1}{2}}]^{-1}\, m_0^{(0)}$,

where $$K = 2B\gamma^2/[(\omega-\omega_0)^2+\gamma^2].$$

It may be readily verified that $\frac{1}{2}b_1(0)$ can never exceed $0{\cdot}17 m_0^{(0)}$. For this case too, therefore, the approximation is justified. If this be so when $kv = 0$, we feel intuitively that the situation should be even more favourable for *all* non-zero velocities: the movement of the atoms has the effect of obliterating the x dependence of the inversion density. In appendix C we show that for a simplified model $[a_1^2(v)+b_1^2(v)]\,(\overline{m}_v)^{-2}$ is a monotonically decreasing function of v, so that the equations support, in this instance, our intuition. From the same simplified model we also find that the Fourier coefficients are decreasing functions of the intensity parameter B for large values of B, so that even when $B \gg 1$, the third term may still be neglected compared to the first. The omission of the term involving $a_1(v)$ both in (22) and in (25) is further indicated by the fact that $a_1(v)$ and the functions it multiplies are antisymmetric in v. Therefore in the integration over all values of v these terms make a relatively small contribution in the range $|kv| \ll \gamma$. We have already seen in (28) that the $a_1(v)$ terms also make very little contribution in the range $|kv| \gg \gamma$.

For all of the above reasons we finally rewrite (22) as

$$\sum_v J_{Rv}(x,t) = \frac{\wp^2\omega_0\mathscr{E}}{\hbar}\sin kx \sum_v \frac{m_v^{(0)}\, L(\omega-kv)\,[\gamma\cos\omega t+(\omega-\omega_0-kv)\sin\omega t]}{1+B\gamma^2[L(\omega+kv)+L(\omega-kv)]}. \tag{30}$$

We now substitute (30) into the E equation, and replace the sum over all v by an integration to obtain

$$\mathscr{E}\sin kx\,[(c^2k^2-\omega^2)\cos\omega t - 2\sigma\omega\sin\omega t]$$
$$= \mathscr{E}\sin kx\,\frac{4\pi\wp^2\omega_0 m^{(0)}\omega}{\sqrt{\pi}\,\hbar u}\int_{-\infty}^{\infty}\frac{\mathrm{d}v\exp\{-v^2/u^2\}\,L(\omega-kv)\,[\gamma\sin\omega t-(\omega-\omega_0-kv)\cos\omega t]}{1+B\gamma^2[L(\omega+kv)+L(\omega-kv)]}. \tag{31}$$

Here we have also introduced the assumed Maxwellian velocity distribution,

$$m_v^{(0)} = m^{(0)}\exp\{-v^2/u^2\}\,\mathrm{d}v/u\pi^{\frac{1}{2}},$$

u being the thermal velocity, and $m^{(0)}$ the total population inversion density; i.e.

$$m^{(0)} = \sum_v m_v^{(0)}.$$

The solutions will be self-consistent if (31) reduces to an identity. For this to occur the coefficients of the x and t dependent terms must balance. This condition for self-consistency may then be summarized by the two dispersion relations

$$ck-\omega = 2\pi\omega_0 m^{(0)}\alpha_1(\xi, B, \epsilon),$$
$$\sigma = 2\pi\omega_0 m^{(0)}\alpha_2(\xi, B, \epsilon), \tag{32}$$

where α_1 and α_2 are the real and imaginary parts, respectively, of a three-parameter complex integral

$$\alpha(\xi, B, \epsilon) = \frac{\wp^2}{\hbar\pi^{\frac{1}{2}}ku}\int_{-\infty}^{+\infty}\frac{\mathrm{d}y\exp\{-y^2\}}{(y-\xi+\mathrm{i}\epsilon)\left\{1+B\epsilon^2\left[\frac{1}{(y-\xi)^2+\epsilon^2}+\frac{1}{(y+\xi)^2+\epsilon^2}\right]\right\}}, \tag{33}$$

with $$\xi = (\omega-\omega_0)/ku, \quad \epsilon = \gamma/ku.$$

If the restriction $\gamma_a = \gamma_b$ were not made, it may be easily verified that the only changes in the final expressions would be $B = \wp^2\mathscr{E}^2/4\hbar^2\gamma^2 \to \wp^2\mathscr{E}^2/4\hbar\gamma_a\gamma_b$ and $\epsilon = \gamma/ku \to \gamma_{ab}/ku$ where γ_a is the arithmetic mean of γ_a and γ_b.

It is evident that the integral α can be considered the complex polarizability of the active medium. With this in mind, each dispersion relation can be cast into a form in which its physical meaning is apparent. We rewrite the first dispersion relation as

$$\omega = ck-2\pi\omega_0 m^{(0)}\alpha_1(\xi, B, \epsilon). \tag{34}$$

Since $\omega_0/ck \approx 1$, and $2\pi m^{(0)}\alpha_1(\xi, B, \epsilon) \ll 1$, this becomes

$$\omega = ck/n_r = cN\pi/n_r L \quad (N = \text{positive integer}), \tag{35}$$

where $n_r \equiv 1+2\pi m^{(0)}\alpha_1(\xi, B, \epsilon)$ can be thought of as the real part of the index of refraction. In this form the first dispersion relation is just the familiar steady-state frequency determining condition.

Similarly, the second dispersion relation written in terms of the imaginary part of the refractive index $n_i \equiv 2\pi m^{(0)}\alpha_2(\xi, B, \epsilon)$

$$\sigma/\omega_0 = n_i \tag{36}$$

is clearly a statement of the fact that in a steady state the gain is equal to the loss.

4. Steady-state operating conditions for standing wave lasers

The dispersion relations, (32), together with the definition of the polarizability α, (33), allow one in principle to determine the steady-state intensity B and mode frequency ω given a knowledge of σ, $m^{(0)}$, ck and the parameters characterizing the transition. A number of well-known results are immediately obvious from these expressions. For one, α_1 is an odd function of ξ and consequently so is $ck-\omega$. In particular, when $\omega = \omega_0$, $\omega = ck$. The imaginary part of α, α_2, is an even function of ξ. In addition, α_2 is always negative. In order that the second dispersion relation be satisfied, therefore, $m^{(0)}$ must also be negative.

In appendix D we outline the manner in which α may be decomposed into a sum of plasma dispersion integrals, whose values are tabulated (Fried & Conte 1961). We also have

carried out a numerical evaluation of the polarizability. However for certain ranges of its three parameters ξ, B and ϵ, the integral α assumes especially simple forms. In this section we discuss these cases.

(1) $B = 0$; *threshold operation*

In this case

$$\frac{\alpha}{\wp^2/\hbar\pi^{\frac{1}{2}}ku} = -\pi\mathscr{I}[w(\xi+\mathrm{i}\epsilon)] - \mathrm{i}\pi\mathscr{R}[w(\xi+\mathrm{i}\epsilon)], \tag{37}$$

where

$$w(z) = \frac{\mathrm{i}}{\pi}\int_{-\infty}^{+\infty} \mathrm{d}t\,(z-t)^{-1}\exp\{-t^2\}.$$

The function $w(z)$ is related to the complex error function (Abramowitz & Stegun 1964) and to the plasma dispersion function (Fried & Conte 1961).

The first dispersion relation gives

$$ck-\omega = -\frac{2\pi\omega_0 m^{(0)}\wp^2}{\hbar\pi^{\frac{1}{2}}ku}\pi\mathscr{I}[w(\xi+\mathrm{i}\epsilon)]. \tag{38}$$

When ξ is positive $\mathscr{I}[w(\xi+\mathrm{i}\epsilon)]$ is also positive and vice versa. Since $m^{(0)}$ is negative, we obtain the expected result that at threshold ω always lies between ck and ω_0. The mode is said to be 'pulled' toward line centre. In the limit of extreme Doppler broadening, $\epsilon \ll 1$, (38) becomes

$$ck-\omega = -\frac{4\pi\omega_0 m^{(0)}\wp^2}{\hbar ku}\exp\{-\xi^2\}\int_0^{\xi}\mathrm{d}t\exp\{t^2\}.$$

With $B = 0$ the second dispersion relation determines the minimum value of $|m^{(0)}|$, $\hat{m}^{(0)}(\xi,\epsilon)$, necessary to sustain steady-state laser oscillation for any value of ξ. This number itself is a minimum when $\xi = 0$. From (32) and (37) we find

$$\hat{m}^{(0)}(0,\epsilon) = \frac{\sigma ku\hbar}{2\pi^{\frac{3}{2}}\wp^2\omega_0\exp\{\epsilon^2\}\operatorname{erfc}(\epsilon)}. \tag{39}$$

In what follows it will often be a convenient to express results in terms of the ratio

$$\eta = |m^{(0)}|/\hat{m}^{(0)}(0,\epsilon).$$

This quantity is called the relative excitation. In the limit of complete Doppler broadening, (39) becomes

$$\hat{m}^{(0)}(0,0) = \frac{\sigma ku\hbar}{2\pi^{\frac{3}{2}}\wp^2\omega_0}, \tag{40}$$

and, at the other extreme, in the limit of complete homogeneous broadening

$$\hat{m}^{(0)}(0,\infty) = \frac{\sigma\gamma\hbar}{2\pi\wp^2\omega_0}. \tag{41}$$

(2) $\omega-\omega_0 = 0$; *mode at line centre*

In this case $\alpha_1 = 0$, and the imaginary part of α is given by

$$\alpha_2 = -\frac{\pi^{\frac{1}{2}}\wp^2}{\hbar ku}\exp\{\Gamma^2\}\operatorname{erfc}(\Gamma)\,(1+2B)^{-\frac{1}{2}}, \tag{42}$$

where $ku\Gamma = \gamma(1+2B)^{\frac{1}{2}}$ plays the role of an effective homogeneous line width. The second dispersion relation determines the mode intensity B for a given value of the population inversion density $m^{(0)}$ and yields as the connexion between B and the relative excitation, η,

$$\eta = (1+2B)^{\frac{1}{2}} \frac{\exp\{-2B\epsilon^2\}\operatorname{erfc}(\epsilon)}{\operatorname{erfc}[\epsilon(1+2B)^{\frac{1}{2}}]}, \tag{43}$$

or

$$F(\Gamma) = \eta F(\epsilon), \tag{44}$$

where

$$F(y) = y\frac{\exp\{-y^2\}}{\operatorname{erfc}(y)}.$$

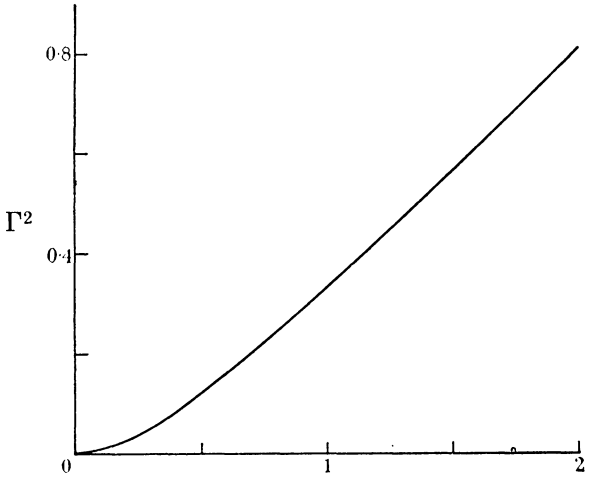

FIGURE 2. The quantity $\Gamma^2 = \epsilon^2(1+2B)$ plotted against $F(\Gamma) = \Gamma \exp\{-\Gamma^2\}/\operatorname{erfc}(\Gamma)$.

In figure 2 we show the result of plotting Γ^2 against $F(\Gamma)$. From this graph one can determine $B(\eta)$ for any value of ϵ. All that is necessary is that the coordinates be relabelled: $B = \frac{1}{2}[(\Gamma^2/\epsilon^2)-1]$, $\eta = F(\Gamma)/F(\epsilon)$. Then that part of the curve lying in the range $B \geqslant 0$, $\eta \geqslant 1$ is the required function. Since the relabelling alters only the scales, for qualitative purposes one can think of the curve in figure 2 immediately as a curve of B against η.

If the line is completely homogeneously broadened, $\epsilon \gg 1$, the mode intensity varies linearly with the relative excitation, at all levels of excitation

$$B = \tfrac{1}{2}(\eta-1). \tag{45}$$

On the other hand, when $\epsilon(1+2B)^{\frac{1}{2}} \ll 1$, (43) gives

$$B = \tfrac{1}{2}(\eta^2-1). \tag{46}$$

With increasing excitation the effective homogeneous line width increases, and the dependence of B on η gradually changes from quadratic to linear.

The experimental results of Smith are in excellent agreement with the predictions of (43), (Smith 1966).

(3) $|\omega-\omega_0| \gg \gamma$; *mode frequency greater than a natural line width from line centre*

Mathematically, this means that the term $(y+\xi)^2+\gamma^2$ can be omitted in the denominator of (33). Physically, this simplification arises when $|\omega-\omega_0| \gg \gamma$, because the atoms in the velocity ranges making the greatest contribution to the integral, $|kv| \approx |\omega-\omega_0|$, 'see' only one travelling wave component of the standing wave in the cavity. (See discussion in § 6.)

Then

$$\alpha_1 = -\frac{\pi^{\frac{1}{2}}\wp^2}{\hbar ku}\mathscr{I}\, w[\xi+\mathrm{i}\epsilon(1+B)^{\frac{1}{2}}], \tag{47}$$

and

$$\alpha_2 = -\frac{\pi^{\frac{1}{2}}\wp^2}{\hbar ku}(1+B)^{-\frac{1}{2}}\mathscr{R}\, w[\xi+\mathrm{i}\epsilon(1+B)^{\frac{1}{2}}]. \tag{48}$$

The effective homogeneous line width here is $\gamma(1+B)^{\frac{1}{2}}$. If this quantity is small compared to ku, then

$$\alpha_1 = -\frac{2\wp^2}{\hbar ku}\exp\{-\xi^2\}\int_0^{\xi} \mathrm{d}t \exp\{t^2\}, \tag{49}$$

and

$$\alpha_2 = -\frac{\pi^{\frac{1}{2}}\wp^2}{\hbar ku}\exp\{-\xi^2\}(1+B)^{-\frac{1}{2}}, \tag{50}$$

and the dispersion relations give

$$ck-\omega = -\frac{4\pi m^{(0)}\omega_0\wp^2}{\hbar ku}\exp\{-\xi^2\}\int_0^{\xi}\mathrm{d}t\exp\{t^2\}, \tag{51}$$

and

$$B = \eta^2\exp\{-2\xi^2\}-1. \tag{52}$$

The frequency of oscillation here too, is 'pulled' toward line centre.

(4) $B \ll 1$, $\epsilon \ll 1$; *operation close to threshold, completely inhomogeneously broadened line*

The restriction on the size of B allows the polarizability to be cast into the form

$$\alpha\Big/\frac{\wp^2}{\hbar\pi^{\frac{1}{2}}ku} = \int_{-\infty}^{+\infty}\frac{\mathrm{d}y\exp\{-y^2\}}{y-\xi+\mathrm{i}\epsilon} - B\epsilon^2\int_{-\infty}^{+\infty}\frac{\mathrm{d}y\exp\{-y^2\}}{y-\xi+\mathrm{i}\epsilon}\left[\frac{1}{(y+\xi)^2+\epsilon^2}+\frac{1}{(y-\xi)+\epsilon^2}\right], \tag{53}$$

and, because of the further stipulation about ϵ, all of the integrations can be performed analytically. The result is, after small terms are neglected,

$$\frac{\alpha_1}{\wp^2/\hbar\pi^{\frac{1}{2}}ku} = -2\pi^{\frac{1}{2}}\exp\{-\xi^2\}\int_0^{\xi}\mathrm{d}t\exp\{t^2\}+\tfrac{1}{2}B\frac{\pi[(\omega-\omega_0)/\gamma]\exp\{-\xi^2\}}{[(\omega-\omega_0)/\gamma]^2+1}; \tag{54}$$

$$\frac{\alpha_2}{\wp^2/\hbar\pi^{\frac{1}{2}}ku} = -\pi\exp\{-\xi^2\}\left[1-\tfrac{1}{2}B\frac{1}{[(\omega-\omega_0)/\gamma]^2+1}-\tfrac{1}{2}B\right]; \tag{55}$$

and

$$ck-\omega = 2\pi\omega_0 m^{(0)}\alpha_1 \tag{56}$$

$$B = 2\left[1+\frac{1}{[(\omega-\omega_0)/\gamma]^2+1}\right]^{-1}(1-\eta^{-1}\exp\{\xi^2\}). \tag{57}$$

This is a case considered in detail by Lamb (1964*a*), Equation (56) agrees with the corresponding one in his paper. In (54) the first term gives rise to frequency 'pulling' and the second to power dependent frequency 'pushing'. The expression for the mode intensity derived here, (57), shows the power tuning dip behaviour predicted by Lamb for certain

values of η in his equation (96). The two equations do differ in that Lamb's intensity is less by a factor $\exp\{-\xi^2\}$, a difference which becomes significant for $|\omega-\omega_0| \geqslant ku$. This falls within the range of validity of (52). For excitation close to threshold, (57) becomes identical to (52). Equation (57) is also consistent with (46) when $\omega = \omega_0$. Thus the two approximations valid for all B in different frequency ranges overlap correctly with (57) which is limited to small B for any ω.

(5) $B \gg 1$ and $B \gg \left|\frac{\omega-\omega_0}{\gamma}\right|$; *operation high above threshold*

$$\alpha_1 = \frac{\wp^2(\omega-\omega_0)}{2\hbar\pi^{\frac{1}{2}}\gamma^2 B}\left[\pi^{\frac{1}{2}} - 2b\pi \exp\{b^2\}\operatorname{erfc}(b)\right], \tag{58}$$

where $$b = (\xi^2+\epsilon^2)^{\frac{1}{2}}$$

and $$\alpha_2 = -\frac{\wp^2}{2\hbar\gamma B}. \tag{59}$$

It is of interest to note that α_2 is, in this case, independent of $\omega-\omega_0$, and as a result, the intensity B is also independent of this quantity. These expressions predict that the Lamb-dip becomes flatter in the region of line centre at high excitation. Uehara & Shimoda (1965) have found by carrying out Lamb's work to one higher order that the Lamb-dip becomes considerably shallower.

(6) $\epsilon \gg 1$; *homogeneously broadened line*

$$\alpha_1 = -\frac{\wp^2}{\hbar\gamma}\frac{(\omega-\omega_0)/\gamma}{[(\omega-\omega_0)/\gamma]^2+2B+1}, \tag{60}$$

$$\alpha_2 = -\frac{\wp^2}{\hbar\gamma}\frac{1}{[(\omega-\omega_0)/\gamma]^2+2B+1}. \tag{61}$$

From the dispersion relations we obtain the well-known results

$$ck-\omega = (\sigma/\gamma)(\omega-\omega_0), \tag{62}$$

or $$\omega = (\sigma+\gamma)^{-1}(\sigma\omega_0+\gamma ck); \tag{63}$$

i.e. the mode frequency is always 'pulled' toward ω_0 and, in addition, is independent of excitation.

Also

$$B = \frac{1}{2}\left[\eta - 1 - \left(\frac{ck-\omega_0}{\sigma+\gamma}\right)^2\right], \tag{64}$$

which of course, reduces to (45) when $ck = \omega_0$.

5. Properties of the population inversion density

Our understanding will be strengthened if we analyse somewhat further the properties of m_v and the physical causes behind them. For the $v = 0$ stream, the equations can be solved, and one obtains for m_v, aside from small time-dependent terms,

$$m_0 = \frac{m_0^{(0)}}{1+4B\gamma^2 L(\omega)\sin^2 kx}. \tag{65}$$

The result has intuitive appeal, for in this case we expect that the intensity acts locally on m as a sink. In (65) the imprint of the assumed standing wave is clearly evident in the x dependence. Figure 3 shows $m_0/m_0^{(0)}$ against x for three values of the coefficient of $\sin^2 kx$ in (65), 0·1, 1·0 and 10.

Similarly, the equations can be handled when $|kv| \gg \gamma$. This is the case in which the atoms considered travel in their lifetime through many wavelengths. Here we expect that the intensity acts as a sink, not locally, but in some average sense, and consequently, that m_v is x independent. Again, the equations support our expectations, the solution being

$$m_v = \frac{m_v^{(0)}}{1+B\gamma^2[L(\omega+kv)+L(\omega-kv)]}. \tag{66}$$

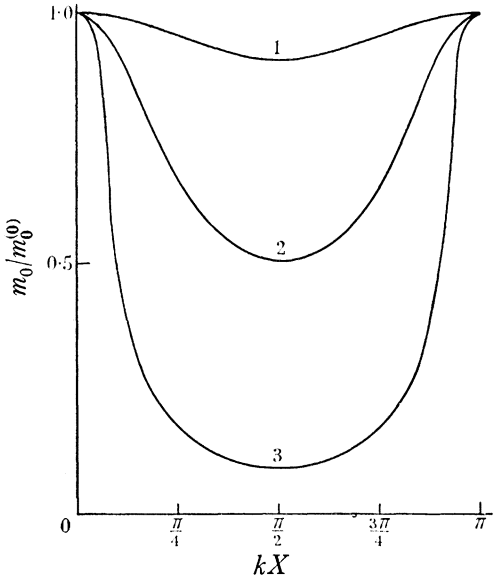

FIGURE 3. $m_0/m_0^{(0)} = (1+R\sin^2 kx)^{-1}$ as a function of kx for three values of R: (1) $R = 0{\cdot}10$, (2) $R = 1{\cdot}0$ and (3) $R = 10$.

We see that the intensity-dependent term in the denominator is just the space average of that in (65). Also we note that both terms in the square brackets in the denominators cannot be large simultaneously. There is a strong similarity between this case and the travelling wave case. If the E field is a travelling wave, m_v is x independent for all v and of the form (see § 6).

$$m_v = \frac{m_v^{(0)}}{1+B\gamma^2 L(\omega-kr)}. \tag{67}$$

This similarity is not merely a coincidence but has a simple physical meaning. In the presence of a standing wave the atoms in the streams for which $|kv| \gg \gamma$ can interact strongly with at most one of the two travelling wave components of the standing wave.

In the intermediate velocity region, we are unable to write an analytical expression for m_v. Certain features of the solution are evident, however. As the velocity of the stream increases, the x dependence of m_v is gradually washed out. Also the appearance of the sine terms,

$$\sum_{n=1}^{\infty} a_n(v) \sin 2nkx,$$

when $kv \neq 0$, in the Fourier expansion of m_v is indicative of the fact that $m_v(x) \neq m_{-v}(x)$, and that, $m_v(x)$ is no longer symmetric in the interval between two nodes or two antinodes of E. We surmise that accompanying the washout of the x dependence there is a shift of the $m_v(x)$ pattern in the direction of the streaming.

If $\gamma/ku \ll 1$ and the laser mode frequency is many natural line widths removed from line centre, i.e. $|\omega-\omega_0| \gg \gamma$, we know m_v against v for all v since in the range $|kv| \leqslant \gamma$ the double integral on the r.h.s. of (23) is small compared to $m_v^{(0)}$. Consequently in this range $m_v = m_v^{(0)}$ and m_v as a function of v gives the familiar 'hole-burning' picture (Bennett 1962*a*, *b*; Lamb 1964*a*).

6. The travelling wave laser

The previous considerations can immediately be adopted to the case when the electric field corresponds to a travelling wave. Indeed, the analysis is much simpler!

We consider again the pair of equations (18) and (19) for m_v and J_v but envisage an E field which corresponds to a travelling wave

$$\begin{aligned} E(x,t) &= 2^{-\frac{1}{2}}\mathscr{E}\sin(\omega t-kx) \\ &= b(x)\exp\{\mathrm{i}\omega t\}+\text{c.c.}, \end{aligned} \tag{68}$$

where

$$b(x) = -\tfrac{1}{2}\mathrm{i}2^{-\frac{1}{2}}\mathscr{E}\exp\{-\mathrm{i}kx\}.$$

Let us transform now to the new variable pair y and t, in place of x and t, where $x = y+vt$. (This corresponds to going into a moving frame of reference, which moves with the same velocity as that of the stream.) J_v, and m_v, and E as functions of y and t shall be denoted $\tilde{J}_v$, $\tilde{m}_v$, and $\tilde{E}$, i.e. $J_v(y+vt,t) = \tilde{J}_v(y,t)$, etc. $\tilde{E}$ can be written as before, with b the same function of y as it was of x, and with ω replaced by $\omega' = \omega-kv$, the Doppler-shifted frequency.

We introduce finally two new functions j_v and k_v by $\tilde{J}_v = bj_v\exp\{\mathrm{i}\omega' t\}+b^*k_v\exp\{-\mathrm{i}\omega' t\}$. This results in the differential equations

$$\left.\begin{aligned} \frac{\partial j_v}{\partial t} &= -\gamma j_v+\mathrm{i}(\omega_0-\omega')j_v+\tilde{m}_v, \\ \frac{\partial k_v}{\partial t} &= -\gamma k_v+\mathrm{i}(\omega_0+\omega')k_v+\tilde{m}_v, \\ \frac{\partial \tilde{m}_v}{\partial t} &= -\gamma(\tilde{m}_v-m_v^{(0)})-[(b^2\exp\{2\mathrm{i}\omega' t\}j_v+|b|^2j_v)+\text{c.c.}] \\ &\quad -[(b^{*2}\exp\{-2\mathrm{i}\omega' t\}k_v+|b|^2k_v)+\text{c.c.}]. \end{aligned}\right\} \tag{69}$$

(For simplicity, in this section the units are such that $2\wp^2\omega_0/\hbar$ and $\frac{1}{2}\hbar\omega_0$ are both unity.)

We notice immediately the following. There is no explicit time dependence in the coefficients of the j_v and k_v equations; j_v will contain a denominator $i(\omega'-\omega_0)-\gamma$, whose absolute value can be small, while k_v will not contain such a small denominator. Thus k_v can be neglected compared to j_v. The $\tilde{m}_v$ equation contains terms which are fast oscillating with the frequency 2ω, and terms which do not contain an explicit time dependence. According to the standard methods developed by Bogoliubov & Mitropolsky (1961), we obtain the zero order approximation by omitting the fast oscillating terms altogether and solving the resulting equations. This set has a time-independent solution. The so resulting $\tilde{m}_v$ is

y independent as well, thus, it is equal to m_v itself. This gives (67), the same expression for m_v as we would obtain by omitting one of the ω dependent terms from the denominator of the standing wave solution. Now we can scale out the field dependence of the complex polarizability by introducing an intensity-dependent line width $\Gamma = \gamma(1+B)^{\frac{1}{2}}$ in place of the original one, γ. In terms of these variables the stationary conditions are the same for all intensities, and hence are the same as at threshold. [For an amplifying medium this result is obtained by Close (1967).]

We may add here that one can also pursue the standing wave case in the same fashion, going to the y, t frame and analysing each pair of equations. In this case, however, the following difference arises. A standing wave contains two plane waves travelling in the opposite directions. Hence, now we obtain two new primed frequencies

$$\omega_1' = \omega - kv, \quad \omega_2' = \omega + kv.$$

Their difference $2kv$, also appears. As long as this difference frequency is large compared to γ nothing is altered, and we can proceed as for the travelling wave case. However, when $2kv$ is of the same order of magnitude as γ, there are no fast oscillating terms left and thus for those streams such that $2kv \lesssim \gamma$ Bogoliubov's method is not applicable any more. Transforming these equations to the x, t frame, we find that this slow t variation gives rise only to an x dependence of m_v. This then constitutes an alternative proof that m_v is indeed primarily time independent, with very small fast-oscillating terms of frequency 2ω, etc.

7. A simplified derivation

Our analysis has suggested that to a good approximation only the t- and x-independent part of m_v will finally contribute to the actual source of the E field in a laser.† If we are willing to accept this we can get to the final results very quickly.

Let us consider (19) with the first of (20) substituted into its r.h.s. We seek that part of m_v which is independent of x and t. Thus we consider m_v a constant and average the equation over t and x; the resulting equation can immediately be solved for the constant part of m_v and gives

$$m_v = m_v^{(0)} \Big/ \left[1 + (4\wp^2/\hbar^2\gamma) \int_0^\infty \mathrm{d}\tau \exp\{-\gamma\tau\} \cos(\omega_0\tau)\, C_v(\tau)\right],$$

where

$$C_v(\tau) = \overline{E(x,t)\, E(x - v\tau, t-\tau)}$$

is the correlation function of the electric field as seen by a particle moving with velocity v through the electromagnetic field.

Evaluating this correlation function for a standing wave we obtain immediately the result of §3. This short cut is remarkable for several reasons. First, the equation resulting in this way from (23) is the same that one would obtain from the standard rate equations involving the Einstein coefficients. Thus the use of rate equations implies an averaging of this kind (Gordon, White & Rigden 1963). Secondly, it suggests that a possible theory could be

† The unexpected feature here is the minor role played by the x dependence of m_v; the t independence is based on deeper and more general reasons. The m_v equation expresses an energy balance in induced emissions. It is well known that for this case a proper energy balance has only been satisfied in the average. In the classical situation this is achieved by averaging the work done by the radiation damping force over several periods; in the quantum mechanical case, it follows from the fact that the commonly used Wigner–Weisskopf approximation is valid only for times large compared to the characteristic period.

developed where only the different correlation functions should appear. Thirdly, it is particularly useful if we intend to include noise in our problem. If this is our intention, we notice that this can be immediately accomplished by decomposing the E field formally into two parts, the noisy part and the rest. Since the two are uncorrelated we obtain the final result immediately by including also the correlation function of the electromagnetic field due to noise (e.g. if thermal radiation of a given temperature would also be present in the laser, we simply add to the correlation function of the electric field generated by the laser, the correlation function of the thermal radiation, and proceed as before). Fourthly, if we desire to include the quantum properties of the electromagnetic field, we must alter this correlation function, which is bilinear in the electric field, according to Klein's rule (Pauli 1933). We do not imply that using this shortcut one can avoid the more detailed analysis, but simply argue that it provides one with a powerful tool to derive quickly heuristic results which may be correct, and probably are.

8. Comparison and outlook

Let us compare our results with those obtained by Lamb (1964*a*). As we have indicated, the fundamental equations are the same in both cases although they appear in rather different forms. Ours is a system of partial differential equations, for E, P_v, and m_v. Lamb's equations on the other hand, are ordinary differential equations for density matrix elements describing the state of the individual two-state atoms. These equations give essentially the characteristics of our partial differential equations for P_v and m_v. Thus, his work can be interpreted as the construction of the solutions for the same set of partial differential equations with the method of characteristics. He does this by expanding the characteristic differential equations in terms of a small parameter, the normalized intensity. We, on the other hand, chose to deal with the partial differential equations themselves without introducing an expansion. This turns out to be quite straightforward for the single-mode case, and we hope that it will be also tractable for the multimode case.

We find that Lamb's first-order and third-order results agree with those obtained here, when the latter are expanded to third order. A fifth-order approximation following Lamb's method has been obtained by Uehara & Shimoda (1965). These various results are compared in figure 4 for the case $\omega = \omega_0$, $\epsilon \ll 1$. From this graph we see that for very small intensities all three results coincide as they should. We also notice that the third-order results deviate less from ours for a larger range of intensities than the fifth-order approximation which already explodes for $B = 0{\cdot}25$. On the other hand the fifth-order curve provides a better fit for very small intensities. From this we surmise that the range of validity of the expansion decreases as one continues the iteration. Lamb also compares his iterative solution with results he obtains heuristically. These heuristic results coincide with ours. Hence one may consider our present work as to provide the theoretical justification for the correctness of the heuristic results.

As far as extensions are concerned there are two directions to discuss: (*a*) What additional problems can be solved by the present equations? (*b*) How can the present equation system and method be improved? (*a*) The present differential equation system specifies an initial value problem. We notice that there are several different time scales in the problem

corresponding to the different frequencies, some of which are large and some of which are small. This suggests strongly that one should subject the differential equations to an asymptotic analysis exploiting the disparities of the time scales, instead of using the smallness of the perturbation as a parameter (Frieman 1963; Sandri 1963). We feel that either the present method, in which we seek only the stationary solutions, or the suggested multiple time-scale expansion will enable us to solve the multimode problem. The present method is already sufficient to treat the single-mode problem with applied external magnetic field. (*b*) The present equation system can be extended to include collisions using the Boltzmann equation to describe the translational motion of the dipoles. One can treat noise and fluctuations by either introducing fictitious currents as sources, according to the method of Lorentz (1916), or by introducing the correlation functions of the electromagnetic field generated by these sources as indicated briefly in § 7.

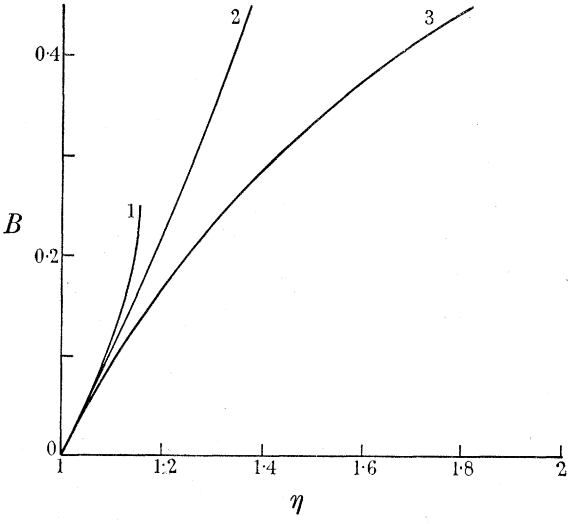

FIGURE 4. The intensity B as a function of the relative excitation η near threshold $\omega = \omega_0$ and $\epsilon \ll 1$. Curve 1, fifth-order results; curve 2, present results; curve 3, third-order results.

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APPENDIX A. LORENTZ AVERAGING

We integrate (5) within a small sphere of volume w centred around the point $\mathbf{r}$, and divide the result by w. The l.h.s. gives immediately the l.h.s. of (6). The r.h.s. gives

$$\sum_j \mathbf{f}_j(\mathbf{R}_j, t)/w,$$

where the sum is extended over all those particles which have the velocity $\mathbf{v}$, and are located within this sphere. Suppose there are l such particles inside, labelled 1, 2, ..., l, the rest being outside. We now compute each term in this sum

$$\mathbf{f}_1(\mathbf{R}_1, t) = \sum_{j \text{ inside}} \mathbf{f}_j(\mathbf{R}_1, t) + \sum_{\alpha \text{ outside}} \mathbf{f}_\alpha(\mathbf{R}_1, t).$$

According to the definition of $\mathbf{f}_1$, $\mathbf{e}_1$ is omitted. The first sum refers to particles inside the sphere, with any velocity, the second sum to those outside, having any velocity. If the particles are isotropically distributed inside, the first sum will give zero. We now compare the second sum with

$$\bar{\mathbf{e}}(\mathbf{R}_1, t) = \sum_{j \text{ inside}} \int_{\mathbf{R}_1} \frac{\mathrm{d}^3\mathbf{r}}{w} \mathbf{e}_j(\mathbf{r}, t) + \sum_{\alpha \text{ outside}} \int_{\mathbf{R}_1} \frac{\mathrm{d}^3\mathbf{r}}{w} \mathbf{e}_\alpha(\mathbf{r}, t),$$

where w is centred on $\mathbf{R}_1$. In the first sum the sources are inside w. There the integration over $\mathbf{e}_j$ will range over the interior and the exterior of the dipoles; the contribution to the integral coming from the exterior part is zero because of the symmetry of the electric field of the dipole.

The integration over the interior of the dipole j gives (with $\mathbf{d}_j = q\mathbf{s}_j$), $-(\mathbf{d}_j/a^3)\,\frac{4}{3}\pi a^3$, if we consider the dipole homogeneous inside and having a radius a. (Actually this is true even more generally.) Upon dividing with w and performing the summation the first term contributes the amount $-\frac{4}{3}\pi n(\mathbf{R}_1, t)\,\mathbf{d}$. In the second term we notice that the sources lie outside w and thus we can approximate $\mathbf{e}_\alpha(\mathbf{r}, t)$ by $\mathbf{e}_\alpha(\mathbf{R}_1, t)$, its value at the centre of the Lorentz sphere. This gives for the second term $\sum_{\alpha \text{ outside}} \mathbf{e}_\alpha(\mathbf{R}_1, t)$ where the summation refers to the exterior particles. By comparison, we find that

$$\begin{aligned}\mathbf{f}_1(\mathbf{R}_1, t) &= \sum_{\alpha \text{ outside}} \mathbf{e}_\alpha(\mathbf{R}_1, t) \\ &= \bar{\mathbf{e}}(\mathbf{R}_1, t) + (\tfrac{4}{3}\pi)\,\bar{\mathbf{p}}(\mathbf{R}_1, t) \\ &= \mathbf{E}(\mathbf{R}_1, t) + (\tfrac{4}{3}\pi)\,\mathbf{P}(\mathbf{R}_1, t),\end{aligned}$$

where the last step follows from the definition of

$$\mathbf{p}(\mathbf{r}, t) = \sum_j q\mathbf{s}_j(t)\,\delta(\mathbf{R}_j - \mathbf{r});$$

thus,

$$\begin{aligned}\bar{\mathbf{p}}(\mathbf{R}_1, t) &= \int \frac{\mathbf{p}(\mathbf{r}, t)\,\mathrm{d}^3\mathbf{r}}{w} \\ &= n(\mathbf{R}_1, t)\,\mathbf{d}.\end{aligned}$$

Observe that we already have assumed that $\mathbf{e}_\alpha(\mathbf{r}, t)$ varies slowly if $\mathbf{r}$ is in w; but then $\mathbf{f}_1(\mathbf{R}_1, t)$ is the same as $\mathbf{f}_2(\mathbf{R}_2, t)$, etc., and we finally find

$$\sum_{\substack{j \text{ in } w \\ \mathbf{R}_j = \mathbf{v}}} \frac{\mathbf{f}_j(\mathbf{R}_j, t)}{w} = n_v(\mathbf{r}, t)\,[\mathbf{E}(\mathbf{r}, t) + \tfrac{4}{3}\pi\mathbf{P}(\mathbf{r}, t)].$$

Appendix B. The dipole moment and population density equations derived from quantum theory

Instead of the Correspondence Principle we can derive our equations following standard techniques. To accept this it is sufficient to derive (7) and (9); the first expresses the equation of motion of the expectation value of the dipole moment of a moving atom in the rest frame of the atom; the second gives the equation of motion for the difference of the probabilities that a moving atom will be in the lower or upper energy states given in the rest frame of the moving atom. This can be obtained immediately as follows. Locate the atom at the origin of its rest frame. In the rest frame express the state of the atom by the density matrix

$$\rho = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ab}^* & \rho_{bb} \end{pmatrix}.$$

The components obey the following differential equations

$$\dot{\rho}_{ab} = -\mathrm{i}\omega_0\rho_{ab} - \gamma\rho_{ab} + \mathrm{i}V(t)\,(\rho_{aa} - \rho_{bb}),$$
$$\dot{\rho}_{aa} = -\gamma\rho_{aa} + \mathrm{i}V(t)\,(\rho_{ab} - \rho_{ba}),$$
$$\dot{\rho}_{bb} = -\gamma\rho_{bb} - \mathrm{i}V(t)\,(\rho_{ab} - \rho_{ba}),$$
$$V = -\wp E(\text{at origin of rest frame})/\hbar.$$

(See Lamb (1964*a*), (25)–(27) with $\gamma_a = \gamma_b$ and with $\omega_0 = (E_a - E_b)/\hbar$.)

The expectation value of the dipole moment is given by $\wp(\rho_{ab} + \text{c.c})$ and the population difference by $\rho_{bb} - \rho_{aa}$. Rediffcrentiating the equation for ρ_{ab} and climinating ρ_{ab} in terms of the dipole moment and its first derivative, we immediately find the sum of (7). The difference of the two equations for $\dot{\rho}_{aa}$ and $\dot{\rho}_{bb}$ gives (9), without the Λ term. The latter is obtained if we add a source term to the equation. (See Lamb (1964*b*) (45).)

Appendix C. Properties of m_v and J_v

Let us consider the pair of equations

$$\mathrm{D}_v J_v/\mathrm{D}t = \mathrm{i}\omega_0 J_v - \gamma J_v + m_v E, \tag{C 1}$$
$$\mathrm{D}_v m_v/\mathrm{D}t = -\gamma(m_v - m_v^{(0)}) - E J_{Rv}. \tag{C 2}$$

Put $J_v = Z_v \exp\{\mathrm{i}\phi_v\}$; Z_v, ϕ_v real. We obtain

$$\left.\begin{aligned} \mathrm{D}_v Z_v/\mathrm{D}t &= -\gamma Z_v + m_v E\cos\phi_v, \\ \mathrm{D}_v \phi_v/\mathrm{D}t &= \omega_0 - m_v E\sin\phi_v/Z_v. \end{aligned}\right\} \tag{C 3}$$

Multiply the Z_v equation with Z_v, the m_v equation with m_v and add. One finds

$$\tfrac{1}{2}(\mathrm{D}_v/\mathrm{D}t)\,(Z_v^2 + m_v^2) = -\gamma Z_v^2 - \gamma(m_v^2 - m_v m_v^{(0)}). \tag{C 4}$$

Average this equation over x and t. The derivative vanishes and we are left with the condition that in a steady state

$$\overline{Z_v^2} + \overline{m_v^2} = \overline{m}_v m_v^{(0)}. \tag{C 5}$$

(If γ is zero $Z_v^2 + m_v^2$ is a constant of the motion along each stream. Its value is proportional to the square of the cooperation number introduced by Dicke (1954).)

From (C 5) property (*b*) follows when one omits the positive quantity $\overline{Z_v^2}$; property (*c*) follows by noticing that the l.h.s. of (C 5) is always positive, unless the solution is identically zero. Property (*a*) can be shown as follows. First we notice that if $m_v^{(0)}$ is zero, Z_v, m_v must be identically zero. Thus the homogeneous equation pair has no solutions. We suppose now that there are two solutions corresponding to the same $m_v^{(0)}$ and take their difference. This difference satisfies the homogenous equations pair, hence the difference must be zero.

Let us write the formal solutions of (C 1) and (C 2) as

$$J_v = \int_0^\infty \mathrm{d}\tau \exp\{-\gamma\tau\} \exp\{\mathrm{i}\omega_0\tau\}\, m_v(x - v\tau, t - \tau)\, E(x - v\tau, t - \tau), \tag{C 6}$$

$$m_v = m_v^{(0)} - \int_0^\infty \mathrm{d}\tau \exp\{-\gamma\tau\} J_{Rv}\,(x - v\tau, t - \tau)\, E(x - v\tau, t - \tau). \tag{C 7}$$

From (C 6) we insert the real part of J_v in (C 2). The resulting integral differential equation (C 8) below requires the symmetry property (*e*).

$$\partial m_v/\partial t + v\,\partial m_v/\partial x = -\gamma(m_v - m_v^{(0)}) - E(x,t)\int_0^\infty \mathrm{d}\tau \exp\{-\gamma\tau\}(\cos\omega_0 t)\, m_v(x-v\tau, t-\tau)\, E(x-v\tau, t-\tau). \quad \text{(C 8)}$$

Let us assume now that m_v is independent of t; from (C 6) we find that J_v must be periodic in t with period $2\pi/\omega$. If J_v has this property, from (C 7) we find that m_v must have a term which is independent of t, and a term which is periodic in t corresponding to the frequency 2ω. The time-independent term cannot be zero, since $\bar{m}_v$ is not, by property (*c*). The other time-dependent term has a small coefficient, since upon integration it acquires a small factor γ/ω; this leads to property (*d*). If m_v is independent of time, it follows from (C 8) that m_v must have the same periodicity as E^2 which is property (*f*). Property (*g*) follows immediately from (C 6) and the properties of m_v.

One considers now the integral-differential equation (C 8) for m_v. We insert

$$E = \mathscr{E}\sin kx\cos\omega t,$$

average it over time (to obliterate the fast oscillations) and seek the solution of the resulting equation in the form

$$m_v = m_v^{(0)}\sum_n C_n(v)\exp\{2\mathrm{i}nkx\}.$$

This leads immediately to the recursion relations

$$(2\mathrm{i}\kappa n+1)\,C_n = \delta_{n,0} - B\sum_\lambda\left[\frac{1}{1+\mathrm{i}\kappa(2n+\lambda)+\mathrm{i}\Omega}+\frac{1}{1+\mathrm{i}\kappa(2n+\lambda)-\mathrm{i}\Omega}\right](C_n - C_{n+\lambda}), \quad \text{(C 9)}$$

where B is a real constant, proportional to $\mathscr{E}^2$; κ is kv/γ, and $\Omega = (\omega-\omega_0)/\gamma$, $\lambda = 1, -1$. Our aim is to understand qualitatively the nature of the solutions. It is important to notice that the coefficients are complex, and decreasing in magnitude as n increases, and also as κ increases. For this reason we approximate these recursion relations with simpler ones, where we keep the coefficients complex but independent of n, though still decreasing functions of κ. In addition, we take $\Omega = 0$. This way we worsen the convergence. The new recursion relations for the new set of complex numbers p_n shall be

$$\left.\begin{aligned} p_n &= -A\exp\{-\mathrm{i}\phi\}p_n + A\exp\{-\mathrm{i}\phi\}\tfrac{1}{2}(p_{n+1}+p_{n-1}) \quad (n>0),\\ p_0 &= 1 - Qp_0 + \tfrac{1}{2}[Q(p_1+p_1^*)] \quad (n=0),\\ p_{-n} &= p_n^*. \end{aligned}\right\} \quad \text{(C 10)}$$

Here

$$A = \frac{4B}{1+2\kappa^2}, \quad Q = \frac{4B}{1+\kappa^2},$$

$$\exp\{-\mathrm{i}\phi\} = \frac{1-2\mathrm{i}\kappa}{1+2\mathrm{i}\kappa}.$$

Put $\zeta = A^{-1}\exp\{\mathrm{i}\phi\}+1$, which gives

$$2p_n\zeta = p_{n+1}+p_{n-1} \quad (n>0). \quad \text{(C 11)}$$

This is immediately soluble in a continued fraction expansion,

$$\frac{p_n}{p_{n-1}} = \frac{1}{2\zeta - (p_{n+1}/p_n)} = \frac{1}{2\zeta - \dfrac{1}{2\zeta - }}. \tag{C12}$$

If we call the continued fraction r, it satisfies

$$r = (2\zeta - r)^{-1} \quad \text{or} \quad r_\pm = \zeta \pm (\zeta^2 - 1)^{\frac{1}{2}}.$$

The product of the roots is unity, hence one of the roots will have absolute value less than unity and this root makes the p_n sequence convergent. The successive ratios p_n/p_{n-1}, are all equal. Knowing the ratio p_0/p_1, we can determine p_0 and p_1 from the recursion relation for $n = 0$. The sequence p_n can be represented by a set of vectors in the complex plane in the following manner. One draws the logarithmic spiral $R = p_0 \exp\{\sigma^{-1}\phi \ln|r|\}$ where R and ϕ are polar coordinates in the plane; $|r|$ is the modulus and σ the phase of r, $r = |r| \exp\{i\sigma\}$. The set of p_n corresponds to radial arrows radiating from the origin, teminating on the spiral, their angular separation from each other being all the same and equal to σ. The ratio of two successive terms is a decreasing function of κ. Since p_0 changes very little with κ, we see that it is p_1 which decreases with κ. As for the A dependence (intensity dependence) we find that for large A's r approaches unity from below and p_0 diminishes. Thus the p's become more and more equal and smaller in magnitude.

Appendix D. Evaluation of the complex polarizability

We wish to analyse the following integral

$$\int_{-\infty}^{+\infty} \frac{dz \exp\{-(z^2/\eta^2)\}}{(z-a+i)\left\{1+B\left[\dfrac{1}{(a+z)^2+1}+\dfrac{1}{(a-z)^2+1}\right]\right\}} = \int_{-\infty}^{+\infty} \frac{dz \exp\{-(z^2/\eta^2)\}}{z-a+i} - \int_{-\infty}^{+\infty} dz \exp\{-(z^2/\eta^2)\} F(z),$$

where

$$F(z) = \frac{G(z)}{1+G(z)},$$

$$G(z) = B\left(\frac{1}{(z-a)^2+1}+\frac{1}{(z+a)^2+1}\right),$$

and where η, B, and a, are real, and the integration is along the real axis. The first integral is tabulated. The second integral can be reduced to a sum of similar tabulated integrals. We write $F(z)$ as a simple fraction. The denominator is a polynomial of fourth degree and has four roots. Since F is real and an even function of z the four roots must be arranged as z_F, $-z_F$, z_F^*, $-z_F^*$, if z_F is one of the roots. Knowing this, one can immediately decompose the second integrand into partial fractions, which lead to five additional integrals, all of the same type; one of them is the same as the first integral, except for a different coefficient. The rest are the complex plasma dispersion integrals, which are tabulated. We have analysed the motion of the roots in the complex plane as a function of the intensity B and distance from the line centre a, hoping that their behaviour may indicate certain interesting regions for these parameters. In particular, for a given distance a from the line centre, there is an intensity $B_1 = 2a^2[1 + a^{-1}(1+a^2)^{\frac{1}{2}}]$ for which the four roots merge into two. Above

this intensity, all four roots are pure imaginary. One would expect that at this field intensity something physically interesting may happen; however, this expectation remains to be gratified.

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